Counting Curves in Elliptic Surfaces by Symplectic Methods

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Abstract

We explicitly compute family GW invariants of elliptic surfaces for primitive classes. That involves establishing a TRR formula and a symplectic sum formula for elliptic surfaces and then determining the GW invariants using an argument from [IP3]. In particular, as in [BL1], these calculations also confirm the well-known Yau-Zaslow Conjecture [YZ] for primitive classes in $K3$ surfaces.

In [L] we introduced "family GW invariants" for Kähler surfaces with $p_g > 0$. Since these invariants are defined by using *non-compact* family of almost Kähler structures, we can easily extend several existing techniques for calculating GW invariants to the family GW invariants. In particular, the 'TRR formula' applies to the family invariants, and at least some special cases of the symplectic sum formula [IP3] apply, with appropriate minor modifications to the formula. Those formulas enable us to enumerate the curves in the elliptic surfaces $E(n)$ for the class $A=$ section plus multiples of the fiber.

Theorem 0.1 *Let $E(n) \to \mathbb{P}^1$ be a standard elliptic surface with a section of self-intersection $-n$. Denote by s and f the homology class of the section and the fiber. Then the genus g family GW invariants for the classes $s + df$ are given by the generating function*

$$\sum_{d \geq 0} GW^{\mathcal{H}}_{s+df,g}(E(n))(pt^g) \, t^d \;=\; (\, tG'(t) \,)^g \prod_{d \geq 1} \left(1 - t^d \right)^{-12n} \qquad (0.1)$$

where $G(t) = \sum\limits_{d \geq 1} \sigma(d) \, t^d$ and $\sigma(d) = \sum_{k|d} k$.

Bryan and Leung ([BL1],[BL2]) defined family invariants for K3 and Abelian surfaces by using the Twistor family. They used algebraic methods to show (0.1) for GW invariants of the rational elliptic surface $E(1)$ and for family invariants of $E(2) = K3$ surfaces. For K3, that confirms the famous Yau-Zaslow Conjecture [YZ] for those cases when the homology class A is primitive. They also pointed out that one can define family invariants of $E(n)$ for $n \geq 3$ using compact family of complex structures induced from the fiber sum, and then use the algebraic methods of [BL1] to show that those invariants also satisfy (0.1) [BL4], see also section 5 of [BL3].

On the other hand, Ionel and Parker used analytic methods to compute the GW invariants of $E(1)$ [IP3]. They related TRR formula and their sum formula for the relative invariants to obtain a quasi-modular form as in (0.1). We follow the same argument — relating TRR formula and sum formula — to show Theorem 0.1. This theorem also confirm the Yau-Zaslow Conjecture for primitive classes, since our invariants of K3 surfaces are equivalent to the invariants define by Bryan and Leung (cf. Theorem 4.3 of [L]).

For $E(1)$ and $E(2)$, the invariants are known to be enumerative, that is, formula (0.1) actually counts (irreducible) holomorphic curves in the primitive classes for generic complex structures on those surfaces [BL1]. At the moment, it is not clear whether, or in what sense, that is true for the $E(n)$ with $n \geq 3$ (cf. Remark 5.12 of [BL3]).

The construction of family invariants for Kähler surfaces is briefly described in Section 1. We give an overview of the proof of Theorem 0.1 in Section 2. This argument is an extension of the elegant argument used by Ionel and Parker to compute the GW invariants of $E(1)$ [IP3]. It involves computing the generating function for the invariants in two ways, first using the so-called TRR formula, and second using a symplectic sum formula as in [IP3]. Roughly, the only modification needed is a shift in the dimension counts. The extended TRR formula is proved in Section 3 and the sum formula are established in the last 5 sections.

Section 4 gives an alternative definition of the family invariants for $E(n)$ based on the idea of perturbing the J_α-holomorphic map equations as in [RT1, RT2]. This alternative definition is better suited to adapt the analytic arguments in [IP2, IP3] to a family version of sum formula. The proof of the sum formula begins by studying holomorphic maps into a degeneration of $E(n)$. Because $E(n)$ is a Kähler surface we are able to degenerate within a holomorphic family, rather than the symplectic family used in [IP3].

The degeneration family Z is described in Section 5. It is a family $\lambda : Z \to D^2$ whose fiber Z_λ at $\lambda \neq 0$ is a copy of $E(n)$ and whose central fiber is a union of $E(n)$ with $E(0) = T^2 \times S^2$ along a fixed elliptic fiber V. As $\lambda \to 0$ maps into Z_λ converge to maps into Z_0, and by bumping α to zero along the fiber V we can ensure that the limits satisfy a simple matching condition along V (there is a single matching condition for the classes A that we consider). Section 6 shows this splitting argument.

Conversely, if a map into Z_0 satisfies the matching condition then it can be smoothed to produce a map into Z_λ for small λ. That smoothing is the Gluing Theorem in [IP3], which relate family invariants of $E(n)$ with relative invariants of $E(n)$ and $E(0)$ relative to V. We define a family version of relative invariants of $E(n)$ in Section 7. Using the Gluing Theorem, we prove the required sum formulas for the family invariants of $E(n)$ in Section 8.

1   Family Invariants for Kähler surfaces

Let X be a closed complex surface with Kähler structure (ω, J, h). In this section we briefly describe the family Gromov-Witten invariants associated to (X, J) which were defined in [L].

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First set

$$\mathcal{H} \;=\; \{\, \alpha + \overline{\alpha} \mid \alpha \in H^{2,0}(X) \,\}.$$

This is a $2\,p_g$-dimensional space of harmonic forms which are J-*anti-invariant*, that is, $\alpha(Ju, Jv) = -\alpha(u,v)$. Each $\alpha \in \mathcal{H}$ defines an endormorphism K_α of TX by the equation

$$h(u, K_\alpha v) \;=\; \alpha(u,v).$$

One can check that, for each $\alpha \in \mathcal{H}$, $Id + JK_\alpha$ is invertible, so defines an almost complex structure

$$J_\alpha \;=\; (\, Id + JK_\alpha \,)^{-1} \, J \, (\, Id + JK_\alpha \,).$$

Let $\overline{\mathcal{F}} = \overline{\mathcal{F}}_{g,k,A}$ the space of all stable maps $f : (C,j) \to X$ of genus g with k marked points which represent homology class A. For each such map, collapsing unstable components of the domain determines a point in the Deligne-Mumford space $\overline{\mathcal{M}}_{g,k}$ and evaluation of marked points determines a point in X^n. Thus we have a map

$$\overline{\mathcal{F}} \xrightarrow{\ st \times ev\ } \overline{\mathcal{M}}_{g,k} \times X^k \tag{1.1}$$

where st and ev denote stabilization map and evaluation maps, respectively. On the other hand, there is a *generalized orbifold bundle* E over $\overline{\mathcal{F}} \times \mathcal{H}$ whose fiber over $(\,f, j, \alpha\,)$ is $\Omega^{0,1}_{j J_\alpha}(f^* TX)$. This bundle has a section Φ defined by

$$\Phi(f, j, \alpha) = df + J_\alpha \, df \, j \,. \tag{1.2}$$

By definition, the right-hand side of (1.2) vanishes for J_α-holomorphic maps. Thus $\Phi^{-1}(0)$ is the moduli space of J_α-holomorphic maps which we denote by

$$\overline{\mathcal{M}}^{J,\mathcal{H}}_{g,k}(\,X, A\,).$$

It is, unfortunately, not always compact. When it is compact, it gives rise to family Gromov-Witten invariants in the usual way (cf. [L]).

Proposition 1.1 ([LT]) *Suppose the moduli space $\Phi^{-1}(0)$ is compact. Then the bundle E has a rational homology "Euler class" $[\overline{\mathcal{M}}^{J,\mathcal{H}}_{g,k}(X,A)]^{\mathrm{vir}} \in H_{2r}(\overline{\mathcal{F}}; \mathbb{Q})$ for*

$$r \;=\; c_1(X)[A] \;+\; g - 1 \;+\; k \;+\; p_g.$$

Definition 1.2 *Whenever the moduli space $\overline{\mathcal{M}}^{J,\mathcal{H}}_{g,k}(\,X, A\,)$ is compact, we define the family GW invariants of (X, J) to be the map*

$$GW^{J,\mathcal{H}}_{g,k}(X, A) \,:\, H^*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q}) \times [\, H^*(\,X; \mathbb{Q}\,)\,]^k \;\to\; \mathbb{Q}$$

defined on $\beta \in H^*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q})$ and $\alpha \in H^*(\,X^k; \mathbb{Q}\,)$ by

$$GW^{J,\mathcal{H}}_{g,k}(X, A)(\beta; \alpha) \;=\; [\overline{\mathcal{M}}^{J,\mathcal{H}}_{g,k}(X, A)]^{\mathrm{vir}} \cap (\, st^*(\beta) \cup ev^*(\alpha)\,).$$

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This paper will focus on the case where X is a standard elliptic surface $E(n)$ with a section class. Note that the elliptic surfaces $E(2)$ are K3 surfaces. Denote by s and f the homology class of the section and the fiber of $E(n)$. Since $c_1(E(n)) = (2-n)f$ and $p_g = n-1$, we have

$$\dim \overline{\mathcal{M}}_{g,k}^{J,\mathcal{H}}(\,E(n), A\,) \;=\; 2(g+k). \tag{1.3}$$

Proposition 1.3 ([L]) *Let (X, J) be an elliptic surface $E(n)$ and $A = s + df$, where d is an integer.*

(a) *The moduli space $\overline{\mathcal{M}}_{g,k}^{J,\mathcal{H}}(X, A)$ is compact and hence the invariants $GW_{g,k}^{J,\mathcal{H}}(X, A)$ are well-defined.*

(b) *The invariants $GW_{g,k}^{J,\mathcal{H}}(X, A)$ depend only on the deformation class of (X, J).*

(c) *For K3 surfaces (i.e. n=2) the $GW_{g,k}^{J,\mathcal{H}}(X, A)$ are same as the invariants defined by Bryan and Leung in [BL1].*

Thus for elliptic surfaces the family of J_α-holomorphic maps parameterized by the family $\mathcal{H}$ gives rise to well-defined invariants, which we will denote variously as

$$GW_{g,k}^{\mathcal{H}}(E(n), A)\,, \quad GW_{A,g}^{\mathcal{H}}(E(n))\,, \quad \text{or simply} \quad GW_{A,g}^{\mathcal{H}}.$$

The goal of this paper is to *calculate* these family GW invariants.

The family invariants have a property analogous to the composition law of ordinary GW invariants. Consider a node p of a stable curve C in the Deligne-Mumford space $\overline{\mathcal{M}}_{g,k}$. When the node is separating, the normalization of C has two components. The genus and the number of marked points decompose as $g = g_1 + g_2$ and $k = k_1 + k_2$ and there is a natural map

$$\sigma : \overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \to \overline{\mathcal{M}}_{g,k}. \tag{1.4}$$

defined by identifying $(k_1 + 1)$-th marked points of the first component to the first marked point of the second component. We denote by $PD(\sigma)$ the Poincaré dual of the image of this map σ. For non-separating node, there is another natural map

$$\theta : \overline{\mathcal{M}}_{g-1,k+2} \to \overline{\mathcal{M}}_{g,k}$$

defined by identifying the last two marked points. We also write $PD(\theta)$ for the Poincaré dual of the image of this map θ.

Proposition 1.4 ([L]) *Let $\{H_\gamma\}$ be any basis of $H^*(X; \mathbb{Z})$ and $\{H^\gamma\}$ be its dual basis and suppose that $GW_{g,k}^{J,\mathcal{H}}(X, A)$ is defined.*

(a) *Given any decomposition $A = A_1 + A_2$, $g = g_1 + g_2$, and $k = k_1 + k_2$, if the moduli space $\overline{\mathcal{M}}_{g_1,k_1}^{J,\mathcal{H}}(\,X, A_1\,)$ is compact, then*

$$GW_{A,g}^{J,\mathcal{H}}(X)(PD(\sigma); \alpha_1, \cdots, \alpha_k)$$
$$= \sum_{A=A_1+A_2} \sum_{\gamma} GW_{A_1,g_1}^{J,\mathcal{H}}(X)(\alpha_1, \cdots, \alpha_{k_1}, H_\gamma)\, GW_{A_2,g_2}(X)(H^\gamma, \alpha_{k_1+1}, \cdots, \alpha_k)$$

where $GW_{A_2,g_2}(X)$ denotes the ordinary GW invariant.

(b) *$GW_{A,g}^{J,\mathcal{H}}(X)(PD(\theta); \alpha_1, \cdots, \alpha_k) = \sum_\gamma GW_{A,g-1}^{J,\mathcal{H}}(X)(\alpha_1, \cdots, \alpha_k, H_\gamma, H^\gamma).$*

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2 The Invariants of $E(n)$ — Outline

By Proposition 1.3, the family GW invariants of $E(n)$ for the class $s + df$ are unchanged under deformations of Kähler structure. Since the moduli space with genus g and no marked points has dimension $2g$, we get numerical invariants by imposing g point constraints on the moduli space $\overline{\mathcal{M}}^{\mathcal{H}}_{g,g}(E(n), s + df)$ — those are the numbers we aim to calculate. For convenience we assemble them in the generating function

$$F_g(t) = \sum_{d \geq 0} GW^{\mathcal{H}}_{s+df,g}(E(n))(pt^g)\, t^d. \tag{2.1}$$

In this and the following four sections we will derive the formula for $F_g(t)$ stated in Theorem 0.1. Thus our aim it to prove:

Proposition 2.1 *For $n \geq 1$,*

$$F_g(t) = (tG'(t))^g \prod_{d \geq 0} \left(1 - t^d\right)^{-12n} \tag{2.2}$$

This section shows how Proposition 2.1 follows from three formulas, equations (2.4), (2.5) and (2.6) below, that are proved in later sections. Our proof parallels the proof of Ionel and Parker for GW invariants of $E(1)$ [IP3].

Here is the outline the proof of (2.2). Consider the 'descendent' $\tau(f^*) = \psi_1 \cup ev^*(f^*)$ where ψ_1 denotes the first Chern class of the line bundle $L \to \overline{\mathcal{M}}^{\mathcal{H}}_{1,1}(E(n), s + df)$ whose geometric fiber over $(f, (C; x), \alpha)$ is $T^*_x C$. We define the generating function for a genus 1 invariant with the descendent constraint, namely

$$H(t) = \sum_{d \geq 0} GW^{\mathcal{H}}_{s+df,1}(E(n))(\tau(f^*))\, t^d. \tag{2.3}$$

We can compute $H(t)$ in two different ways. In section 3, we show how to combine the composition law together with the TRR for genus 1 to obtain the formula

$$H(t) = \frac{1}{12}\, t\, F_0'(t) - \frac{1}{12}\, F_0(t) + (2 - n)\, F_0(t)\, G(t) \tag{2.4}$$

Then, from section 4 to 8 we establish a family version of the sum formulas to show

$$H(t) = -\frac{1}{12}\, F_0(t) + 2\, F_0(t)\, G(t) \tag{2.5}$$

$$F_g(t) = F_{g-1}(t)\, t\, G'(t) \tag{2.6}$$

(see Proposition 8.5). Equations (2.4) and (2.5) give rise to the ODE

$$t\, F_0'(t) = 12\, n\, G(t)\, F_0(t) \tag{2.7}$$

and we show in Proposition 4.4 that the initial condition is $F_0(0) = 1$. It is well-known that the solution of this ODE is given by

$$F_0(t) = \prod_{d \geq 0} \left(1 - t^d\right)^{-12n}.$$

Now, (2.6) gives (2.2) by induction. That completes the proof of Proposition 2.1 and hence of the main Theorem 0.1 of the introduction. The heart of the matter, then, is to establish formulas (2.4), (2.5) and (2.6).

3 The Topological Recursion Relation (TRR)

This section shows the TRR formula (2.4). Following [AC], we denote by $\mathcal{M}(G)$ the moduli space of all genus g stable curves with k marked points whose dual graph is G. We also denote by δ_G the orbifold fundamental class of $\overline{\mathcal{M}}(G)$, that is, the fundamental class divided by the order of the automorphisms of a general element of $\mathcal{M}(G)$. Graphs with one edge correspond to degree two classes. There are two types of such graphs, one of which is the graph G_{irr} with one vertex of genus $g - 1$.

The following is the well-known genus 1 topological recursion relation:

$$\phi_1\ (= c_1(\mathcal{L}))\ =\ \frac{1}{12}\,\delta_{G_{irr}} \quad \text{in} \quad H^2(\overline{\mathcal{M}}_{1,1}; \mathbb{Q}) \tag{3.1}$$

where the line bundle $\mathcal{L} \to \overline{\mathcal{M}}_{1,1}$ has the geometric fiber $T_x^* C$ at the point (C, x).

Proposition 3.1 *The generating function (2.3) satisfies*

$$H(t)\ =\ \frac{1}{12}\,t\,F_0'(t)\ -\ \frac{1}{12}\,F_0(t)\ +\ (2 - n)\,F_0(t)\,G(t).$$

Proof. Let $\overline{\mathcal{M}}_{0,2}$ be the space of prestable curves of genus 0 with two marked points [G] and $\sigma : \overline{\mathcal{M}}_{0,2} \times \overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ be the gluing map as in (1.4). For any decomposition of $s + df = A_1 + A_2$, we denote by

$$\overline{\mathcal{M}}(\sigma(A_1, A_2))\ \subset\ \overline{\mathcal{M}}_{1,1}^{\mathcal{H}}(s + df) \tag{3.2}$$

the set of all (f, C, α) in $\overline{\mathcal{M}}_{1,1}^{\mathcal{H}}(A)$ such that (i) $C = \sigma(C_1, C_2)$ for some $C_1 \in \overline{\mathcal{M}}_{0,2}$ and $C_2 \in \overline{\mathcal{M}}_{1,1}$, (ii) the restriction of f to C_1 represents A_1, and (iii) the restriction of f to C_2 represents A_2. By the machinery of Li and Tian [LT], there is a virtual fundamental class

$$[\overline{\mathcal{M}}(\sigma(A_1, A_2))]^{\mathrm{vir}} \tag{3.3}$$

associated with (3.2) such that the coefficients $GW_{s+df,1}^{\mathcal{H}}(\tau(f^*))$ of $H(t)$ are

$$[\overline{\mathcal{M}}_{1,1}^{\mathcal{H}}(s + df)]^{\mathrm{vir}}\ \cap\ \tau(f^*)\ =\ [\overline{\mathcal{M}}_{1,1}^{\mathcal{H}}(s + df)]^{\mathrm{vir}}\ \cap\ (st^*\phi_1 \cup ev^*(f^*))$$
$$+ \sum [\overline{\mathcal{M}}(\sigma(A_1, A_2))]^{\mathrm{vir}}\ \cap\ ev^*(f^*) \tag{3.4}$$

where the sum is over all decompositions $s + df = A_1 + A_2$.

Let $\{H^\gamma\}$ and $\{H_\gamma\}$ be bases of $H^*(E(n))$ dual by the intersection form. We have

$$[\overline{\mathcal{M}}_{1,1}^{\mathcal{H}}(s + df)]^{\mathrm{vir}}\ \cap\ (st^*\phi_1 \cup ev^*(f^*))\ =\ \frac{1}{12}\,GW_{s+df,1}^{\mathcal{H}}(\delta_{G_{irr}}; f^*)$$
$$=\ \frac{1}{24} \sum_\gamma GW_{s+df,0}^{\mathcal{H}}(f^*, H^\gamma, H_\gamma)$$
$$=\ \frac{2d - n}{24}\,GW_{s+df,0}^{\mathcal{H}} \tag{3.5}$$

where the first equality follows from (3.1), the second follows from Proposition 1.4 b and $|\operatorname{Aut}(G_{irr})| = 2$, and the last follows from $\sum_\gamma (H^\gamma \cdot A)(H_\gamma \cdot A) = A^2$.

On the other hand, by Theorem 2.4 of [L], every (f, α) in $\overline{\mathcal{M}}_{1,1}^{\mathcal{H}}(s + df)$ has $\alpha = 0$, i.e. f is truly holomorphic. The inclusion (3.2) thus means that the only possible decompositions of $s + df$ with nontrivial virtual class (3.3) are $s + d_1 f$ and $d_2 f$ with $d_1 + d_2 = d$, and $d_1, d_2 \geq 0$.

Proposition 1.4 a and routine dimension counts then imply that

$$\sum [\overline{\mathcal{M}}(\sigma(A_1, A_2))]^{\mathrm{vir}} \cap ev^*(f^*) = \sum_{d_1 + d_2} \sum_{\gamma} GW_{s+d_1 f}^{\mathcal{H}}(f^*, H^\gamma) \, GW_{d_2 f, 1}(H_\gamma). \tag{3.6}$$

This can be further simplified by separating the $d_2 = 0$ term and simplifying using the facts (a) $\sum_\gamma (H^\gamma \cdot A)(H_\gamma \cdot B) = A \cdot B$, (b) $d_2 \, GW_{d_2 f, 1} = (2 - n)\sigma(d_2)$ (see [IP1]), and (c) $GW_{0,1}(H_\gamma) = \frac{1}{24}(K \cdot H_\gamma)$ where $K = (n - 2)f$ is the canonical class (see 1.4.1 Proposition of [KM1]). The righthand side of (3.6) then becomes

$$(2 - n) \sum_{1 \leq d_2 \leq d} GW_{s+d_1 f, 0}^{\mathcal{H}} \sigma(d_2) + \frac{n - 2}{24} GW_{s+df, 0}^{\mathcal{H}} \tag{3.7}$$

The proof now follows from (3.4), (3.5), (3.7) and the definitions of $F_0(t)$ and $H(t)$. $\qquad \square$

4 Ruan-Tian Invariants of $E(n)$

Instead of constructing virtual fundamental class directly from the moduli space of stable J-holomorphic maps, Ruan and Tian [RT1, RT2] perturbed J-holomorphic equation to $\overline{\partial} f = \nu$ where the inhomogeneous term ν can be chosen generically. For generic (J, ν), the moduli space of stable (J, ν)-holomorphic maps is then a compact smooth orbifold with all lower strata having codimension at least two. Ruan and Tian defined GW invariants from this (perturbed) moduli space.

We can follow as similar procedure for the family invariants by introducing an inhomogeneous term into the J_α-holomorphic equation and vary ν. This alternative definition of invariants is more geometric. In particular, using this definition of invariants we can follow the analytic arguments of Ionel and Parker in [IP2, IP3] to show sum formulas (2.5) and (2.6) for the case at hand: the class $s + df$ in $E(n)$.

To simplify notation in this section we will set $A = s + df$.

Using Prym structures defined as in [Lo], we can lift the Deligne-Mumford space $\overline{\mathcal{M}}_{g,k}$ to a finite cover

$$p_\mu : \overline{\mathcal{M}}_{g,k}^\mu \to \overline{\mathcal{M}}_{g,k}. \tag{4.1}$$

This finite cover is now a smooth manifold and has a universal family

$$\pi_\mu : \overline{\mathcal{U}}_{g,k}^\mu \to \overline{\mathcal{M}}_{g,k}^\mu$$

which is projective. Moreover, for each $b \in \overline{\mathcal{M}}_{g,k}^\mu$, $\pi_\mu^{-1}(b)$ is a stable curve isomorphic to $p_\mu(b)$.

We fix, once and for all, an embedding of $\overline{\mathcal{U}}_{g,k}^{\mu}$ into some $\mathbb{P}^N$. An inhomogeneous term ν is then defined as a section of the bundle $\mathrm{Hom}(\pi_1^*(T\mathbb{P}^N), \pi_2^* TE(n))$ which is anti-J-linear :

$$\nu(j_{\mathbb{P}}(v)) = -J(\nu(v)) \qquad \text{for any } v \in T\mathbb{P}^N \tag{4.2}$$

where $j_{\mathbb{P}}$ is the complex structure on $\mathbb{P}^N$.

For each stable map $f : C \to E(n)$, we can specify one element $j \in p_\mu^{-1}(st(C))$. Then $\pi_\mu^{-1}(j)$ is isomorphic to the stable curve $st(C)$. In this way, we can define a map

$$\phi : C \to st(C) \cong \pi_\mu^{-1}(b) \subset \overline{\mathcal{U}}_{g,k}^{\mu} \hookrightarrow \mathbb{P}^N. \tag{4.3}$$

Definition 4.1 *A stable (J, ν, α)-holomorphic map is a stable map $f : (C, \phi) \to E(n)$ satisfying*

$$(df + J_\alpha df j_C)(p) \; = \; \nu_\alpha(\phi(p), f(p))$$

where ϕ is defined as in (4.3), and $\nu_\alpha = (I + JK_\alpha)^{-1}\nu$.

We denote the moduli space of stable (J, α, ν)-holomorphic maps $((f, (\phi, C), \alpha)$ by

$$\overline{\mathcal{M}}_{g,k}(E(n), A, \nu, \mathcal{H}, \mu) \tag{4.4}$$

where α in $\mathcal{H}$ and $[f(C)] = A$ in $H_2(E(n); Z)$. We also denote by

$$\mathcal{M}_{g,k}(E(n), A, \nu, \mathcal{H}, \mu)$$

the set of $((f, (\phi, C)), \alpha)$ with a smooth domain C. We will often abuse notation by writing (f, C, α), (f, j, α) or simply (f, α), instead of $(f, (\phi, C), \alpha)$.

There are stabilization and evaluation maps as in (1.1):

$$\mathcal{M}_{g,k}(E(n), A, \nu, \mathcal{H}, \mu) \xrightarrow{st^\mu \times ev^\mu} \overline{\mathcal{M}}_{g,k}^{\mu} \times E(n)^k. \tag{4.5}$$

Its Frontier is defined to be the set

$$\{ r \in \overline{\mathcal{M}}_{g,k}^{\mu} \times E(n)^k \,|\, r = \lim(st^\mu \times ev^\mu)(f_n, \alpha_n) \text{ and } (f_n, \alpha_n) \text{ has no convergent subsequence} \}.$$

We denote by $\mathcal{Y}_0$ the space of all ν with $|\nu|_\infty$ is sufficiently small. The following is the "Structure Theorem" for the moduli space.

Theorem 4.2 (Structure Theorem) *For generic $\nu \in \mathcal{Y}_0$, the space $\mathcal{M}_{g,k}(E(n), A, \nu, \mathcal{H}, \mu)$ is an smooth oriented manifold of dimension*

$$2\, c_1(A) \; + \; 2\,(g-1) \; + \; 2\,k \; + \; \dim(\mathcal{H}) \; = \; 2\,(g+k). \tag{4.6}$$

Furthermore, the frontier of the smooth map (4.5) lies in dimension at most 2 less than $2(g+k)$.

Sketch of Proof. The proof of this theorem is similar to that of Proposition 2.3 in [RT2]. The first statement follows from the standard argument using Sard-Smale Theorem. To prove the second statement, we first consider the well-defined stabilization and evaluation map

$$\overline{\mathcal{M}}_{g,k}(E(n), A, \nu, \mathcal{H}, \mu) \xrightarrow{st^\mu \times ev^\mu} \overline{\mathcal{M}}_{g,k}^\mu \times E(n)^k. \tag{4.7}$$

It then follows from Gromov Convergence Theorem [IS, P, PW] and Lemma 6.2 below that the stable moduli space (4.4) is compact and hence (4.5) extends (4.7) continuously.

As in [RT1, RT2], we reduce the moduli space by (i) collapsing all ghost bubbles, (ii) replacing each multiple maps from a bubble by its reduced map, and (iii) identifying those bubble components which have the same image. We denote this reduced moduli space by

$$\overline{\mathcal{M}}_{g,k}^r(E(n), A, \nu, \mathcal{H}, \mu).$$

The map (4.5) now descends to the reduced moduli space and by definition we have

$$\mathrm{Fr}(st^\mu \times ev^\mu) \subset st^\mu \times ev^\mu \left(\overline{\mathcal{M}}_{g,k}^r(E(n), A, \nu, \mathcal{H}, \mu) \setminus \mathcal{M}_{g,k}(E(n), A, \nu, \mathcal{H}, \mu) \right).$$

It remains to show that those strata consisting of (f, α) with domain more than two components has a dimension at least 2 less than $2(g + k)$. Similarly to the moduli space of (J, ν)-holomorphic maps, the strata corresponding to the domain with no bubble component has a dimension at least 2 less than $2(g + k)$ for generic ν.

On the other hand, it follows from compactness of stable moduli space (4.4) and Theorem 2.4 of [L] that the restriction of (f, α) to any component of domain should represents one of the following homology classes

$$s, \quad s + d_1 f, \quad d_2 f \quad \text{with} \quad 0 < d_1, d_2 \leq d$$

Since the inhomogeneous term ν vanishes on bubble components, by Theorem 2.4 of [L] that each bubble component maps into either a section or a singular fiber.

Now, suppose (f, α) has some bubble components. Again by Theorem 2.4 of [L] either $\alpha \equiv 0$ or the zero divisor $Z(\alpha)$ contains some singular fibers. Since there's no fixed component in the complete linear system of a canonical divisor of $E(n)$, the parameter α lies in the proper subspace of $\mathcal{H}$. This reduces the dimension of the strata containing (f, α) at least 2. □

Now, we are ready to define invariants. Instead of using intersection theory as in [RT1, RT2], we will follow the approach in [IP2]. The above Structure Theorem and Proposition 4.2 of [KM2] assert that the image

$$st^\mu \times ev^\mu(\mathcal{M}_{g,k}(E(n), A, \nu, \mathcal{H}, \mu))$$

gives rise to a rational homology class in $H_*(\overline{\mathcal{M}}_{g,k}^\mu; \mathbb{Q}) \otimes H_*(E(n)^k; \mathbb{Q})$. We denote it by

$$[\overline{\mathcal{M}}_{g,k}(E(n), A, \nu, \mathcal{H}, \mu)]. \tag{4.8}$$

Definition 4.3 *For $2g + k \geq 3$, we define invariants by*

$$GW_{g,k}(E(n), A, \mathcal{H})(\beta; \alpha) = \frac{1}{\lambda_\mu}(\beta \otimes \alpha) \cap [\overline{\mathcal{M}}_{g,k}(E(n), A, \nu, \mathcal{H}, \mu)]$$

where β in $H^*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q})$, α in $H^*(E(n)^k; \mathbb{Q})$, and λ_μ is the order of the finite cover in (4.1).

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By repeating the same arguments for ordinary GW invariants, we can show that these invariants are same as the family invariants defined in Definition 1.2, namely

$$GW_{g,k}(E(n), A, \mathcal{H}) = GW_{g,k}^{\mathcal{H}}(E(n), A).$$

In the below, we will not distinguish two invariants and use the same notation $GW_{g,k}^{\mathcal{H}}(E(n), A)$ for them. We end this section by showing $F_0(0) = 1$ which provides the initial condition for (2.7).

Proposition 4.4 $GW_{s,0}^{\mathcal{H}}(E(n))(f^3) = 1$.

Proof. Fix $\nu = 0$. Since the section class s is of type $(1, 1)$, Theorem 2.4 of [L] implies that for any (J, α)-holomorphic map (f, α) with $[f] = s$, f is holomorphic and $\alpha = 0$. In fact, there is a unique such f since $s^2 = -n$.

Now, consider the linearization of (f, α)-holomorphic equation $L_f \oplus J\, df \oplus L_0$ as in appendix of [L]. Propositions A.1 and A.2 of the appendix of [L] show, quite generally, that L_f is a $\bar{\partial}$ operator and L_0 defines a map

$$L_0 : \mathcal{H} \to \text{Coker}(L_f \oplus J\, df)$$

which is injective if and only if the family moduli space $\overline{\mathcal{M}}_{g,k}^{\mathcal{H}}(E(n), A)$ is compact. But we just showed the moduli space is a single point, and hence compact.

On the other hand, $\text{Ker}(L_f \oplus Jdf)$ is same as $H^0(f^*N)$, where N is the normal bundle of the section in $E(n)$. It is trivial since the Chern number of N is $s \cdot s = -n < 0$. Therefore,

$$\dim \text{Coker}(\, L_f \oplus Jdf\,) \;=\; -\text{Index}(L_f \oplus Jdf) \;=\; -2\,(\, c_1(f^*TE(n)) - 1\,) \;=\; 2\,(n-1)$$

Since L_0 is injective and $\dim(\mathcal{H}) = 2(n-1)$, $L_f \oplus Jdf \oplus L_0$ is onto. That implies $\nu = 0$ is generic in the sense of Theorem 4.2. Consequently, the invariant is ± 1. In this case, the sign is determined by L_f and L_f is $\bar{\partial}$-operator, the invariant is 1. $\square$

5   Degeneration of $E(n)$

In this section, we describe a degeneration of $E(n)$ into a singular surface which is a union of $E(n)$ and $E(0)$ with $V = T^2$ intersection. We then define the parameter space and inhomogeneous terms corresponding to this degeneration. The sum formulas (2.5) and (2.6) will be formulated from this degeneration

Let $D \subset \mathbb{C}$ be a small disk and choose a smooth fibre V in $E(2)$. We denote by

$$p : Z \to E(n) \times D \tag{5.1}$$

the blow-up of $E(n) \times D$ along $V \times \{0\}$ and define $\lambda : Z \xrightarrow{p} E(n) \times D \to D$ to be the composition map, where the second map is the projection onto the second factor. The central fiber $Z_0 = \lambda^{-1}(0)$ is a singular surface $E(n) \cup_V E(0)$ and the fiber Z_λ with $\lambda \neq 0$ is isomorphic to $E(n)$ as a complex surface.

To save notation, we will use the same notation (ω, J, g) for the induced Kähler structure on Z and its restriction to Z_λ, $E(n)$, and $E(0)$.

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Fix a normal neighborhood $N_{E(n)}$ of V in $E(n)$. It is then a product $V \times D'$, where $D' \subset \mathbb{C}$ is some disk. Let x be the holomorphic coordinate of D'. Then, the normal neighborhood N of V in Z is given by

$$ N \;=\; \{\, (v, x, \lambda, [l_0; l_1]) \mid v \in V,\; x\, l_1 = \lambda\, l_0 \,\} \;\subset\; N_{E(n)} \times D' \times D \times \mathbb{CP}^1 $$

where $[l_0; l_1]$ is the homogeneous coordinates of $\mathbb{CP}^1$. It is covered by two patches $U_0 = (l_0 \neq 0)$ and $U_1 = (l_1 \neq 0)$. On U_0, we set $y = l_1/l_0$. Then we have

$$ N \;=\; \{\, (v, x, y) \mid v \in V \,\} \quad \text{with} \quad \lambda(v, x, y) = xy. $$

Clearly, $Z_\lambda \cap N$ is given by the equation $xy = \lambda$. Note that we can also think of y as a holomorphic normal coordinate of the normal neighborhood $N_{E(0)}$ of V in $E(0)$.

Definition 5.1 *For some $\delta > 0$ and $|\lambda|$, we decompose Z_λ as a union of three pieces, two sides and a neck. The δ-neck is defined as*

$$ Z_\lambda(\delta) \;=\; \Big\{\, (v, x, y) \,\in\, Z_\lambda \cap N \mid \, \big|\, |x|^2 - |y|^2 \,\big| \;\leq\; \delta \,\Big\}. $$

$Z_\lambda \backslash Z_\lambda(\delta)$ *consists of two components. The $E(n)$-side is the component which contains the region* $|x| > |y|$, *while the component $E(0)$-side contains the region* $|x| < |y|$.

On the neck region, there is a symplectic S^1-action with Hamiltonian $t = \frac{1}{2}(\,|y|^2 - |x|^2)$. We can thus decompose each Z_λ as $Z_\lambda = Z_\lambda^- \cup Z_\lambda^+$, where Z_λ^- is a union of $E(n)$-side and the part of $Z_\lambda(\delta)$ with $t \leq 0$. In fact, $E(n)$ (resp. $E(0)$) is the symplectic cut of Z_λ^- (resp. Z_λ^+) at $t = 0$. Therefore, we have a collapsing map

$$ \pi_\lambda : Z_\lambda \to Z_0 \tag{5.2} $$

(cf. section 2 of [IP3]).

Next, we define the parameter spaces. Let U be a neighborhood of V in $E(n)$ that does not contain any singular fibers. Choose a bump function β which satisfies $\beta = 1$ on $E(n) \setminus U$ and $\beta = 0$ near V in U.

Definition 5.2 *We define the parameter spaces by*

$$ \mathcal{H}_\lambda \;=\; \{\, \alpha_\lambda = p_\lambda^* \beta\, \alpha \mid \alpha \in \mathcal{H} \,\} \quad and \quad \mathcal{H}_{E(n)} \;=\; \{\, \beta\, \alpha \mid \alpha \in \mathcal{H} \,\} $$

where p_λ is the restriction of (5.1) to Z_λ.

Note that $\mathcal{H}_{E(n)} = \{0\}$ for $n = 0, 1$. On the other hand, each $\alpha \in \mathcal{H}_{E(n)}$ (resp. $\alpha_\lambda \in \mathcal{H}_\lambda$) is J-anti-invariant and $\alpha = 0$ (resp. $\alpha_\lambda = 0$) near V by definition. Hence $J_\alpha = J$ (resp. $J_{\alpha_\lambda} = J$) near V.

Lastly, following [IP3], we define inhomogeneous terms. An inhomogeneous term ν of the the fibration $\lambda : Z \to D$ is a section of the bundle $\mathrm{Hom}(T\mathbb{P}^N, TZ)$ over $\mathbb{P}^N \times Z$ for some $\mathbb{P}^N$, which satisfies Definition 2.2 of [IP3]. We denote by $\mathcal{J}_0(Z)$ the space of all such ν with sufficiently small $|\nu|_\infty$ and use the same notation ν for the restriction of ν to Z_λ, $E(n)$, and $E(0)$.

6 Splitting of Maps

In this section, we show the uniform energy bound of maps and L^2-bound of parameters α. By Gromov Convergence Theorem, these leads to the compactness of family moduli spaces. The splitting arguments as in section 3 of [IP3] then follows from the compactness and the choice of inhomogeneous terms and parameters α — we bumped α to 0 along V.

Let (X, ω, h, J) be a 4-dimensional almost Kähler manifold. Recall that α is a J-anti-invariant 2-form on X if $\alpha(Ju, Jv) = -\alpha(u, v)$. Fix a metric within the conformal class j on a Riemann surface (C, j) and let dv be the associated volume form.

Lemma 6.1 *For any C^1 map $f : C \to X$ if α is J-anti-invariants, then*

$$f^*\alpha \ \leq \ 2\,|\alpha|\,|df|\,|\overline{\partial}_J f|. \tag{6.1}$$

On the other hand, if the map f is (J, ν, α)-holomorphic, then we have

$$|\overline{\partial}_J f|^2\, dv \ = \ f^*\alpha + 2\,\langle \overline{\partial}_J f, \nu \rangle\, dv \tag{6.2}$$

$$(1 + f^*(|\alpha|^2))\, f^*\omega \ = \ \frac{1}{2}\,(1 - f^*(|\alpha|^2))\,|df|^2\, dv - 4\,\langle \overline{\partial}_J f, \nu \rangle\, dv + 4|\nu|^2\, dv. \tag{6.3}$$

Proof. The proof of (6.2) and (6.3) is similar to those of Corollary 1.4 of [L]. We will prove (6.1) only. Fix a point $z \in C$ and an orthogonal basis $\{e_1, e_2 = je_1\}$ of $T_z C$. Then we have

$$\begin{aligned}
\alpha(\,df(e_1), df(e_2)\,) \ &= \ \alpha(\,df(e_1), df(e_2) + Jdf(je_2) - Jdf(je_2)\,) \\
&= \ \alpha(\,df(e_1), 2\overline{\partial}_J f(e_2)\,) \ + \ \alpha(\,df(e_1), Jdf(e_1)\,).
\end{aligned} \tag{6.4}$$

Since α is J-anti-invariant, $\alpha(\,df(e_1), Jdf(e_1)\,) = 0$. Therefore, (6.1) follows from (6.4). □

We denote the stable family moduli space of (J, ν, α_λ)-holomorphic maps (f, α_λ) by

$$\overline{\mathcal{M}}_{g,k}(\,Z_\lambda, s + df, \nu, \mathcal{H}_\lambda\,), \quad \text{or simply} \quad \overline{\mathcal{M}}_{g,k}(\lambda, d\,)$$

where $\nu \in \mathcal{J}_0(Z)$ and $\alpha_\lambda \in \mathcal{H}_\lambda$. Compactness of the family moduli space follows from Gromov Convergence Theorem and the following lemma.

Lemma 6.2 *Let $|\nu|_\infty$ be sufficiently small. Then, there exit uniform constants E_d and N, which does not depend on λ, such that*

$$E(f) \ = \ \frac{1}{2}\int_C |df|^2 \ \leq \ E_d \quad and \quad ||\alpha_\lambda|| \ = \ \int_{Z_\lambda} \alpha_\lambda \wedge \alpha_\lambda \ \leq \ N$$

for any (f, C, α_λ) in $\overline{\mathcal{M}}_{g,k}(\lambda, d\,)$.

Proof. We first show uniform bound of $||\alpha_\lambda||$. This proof is similar to those of Lemma 4.4 except for using (6.3) instead of Corollary 1.4b of [L]. For each α_λ in $\mathcal{H}_\lambda$, we choose a sufficiently small neighborhood of $N(\alpha_\lambda)$ of the zero set of α_λ and let $m(J_\lambda)$ and N be as in the proof of Lemma 4.4 of [L]. If there is a holomorphic fiber $F \subset E(n) \setminus N(\alpha_\lambda)$ such that

(i) f is transversal to F,

(ii) at each $p \in f^{-1}(F)$, f is transversal to a holomorphic disk $D_{f(p)}$ normal to F at $f(p)$, and

(iii) $4\,|df|\,|\nu| \,+\, 4\,|\nu|^2 \,\leq\, \frac{1}{2}|df|^2$ on $f^{-1}(F)$

then the proof follows exactly as in the proof of Lemma Lemma 4.4 of [L]. We can clearly find fibers satisfying (i) and (ii), so we need only verify that we can also obtain (iii). For that we consider the set C_0 of all points in C where $4\,|df|\,|\nu| + 4|\nu|^2 \,>\, \frac{1}{2}|df|^2$. Then $|df|^2 \leq 100|\nu|^2$ on C_0. Therefore

$$\int_{C_0} |\,d\pi \circ df\,|^2 \;\leq\; 100\,\mathrm{Area}(\,st(C)\,)\,|\,d\pi\,|_\infty^2\,|\,\nu\,|_\infty^2 \tag{6.5}$$

where $\pi : Z_\lambda \to \mathbb{CP}^1$ is the elliptic structure for J on Z_λ. We can thus assume that the left hand side of (6.5) is less than $\frac{1}{3}\mathrm{Area}(\mathbb{CP}^1)$ for sufficiently small $|\nu|_\infty$. On the other hand, from the definition of $N(\alpha_\lambda)$, we can also assume that $\mathrm{Area}(\,\pi(N(\alpha_\lambda))\,) \leq \frac{1}{3}\mathrm{Area}(\mathbb{CP}^1)$. Therefore, we can always choose a holomorphic fiber $F = \pi^{-1}(q)$ as in the above claim with $q \in \mathbb{CP}^1 \setminus (\,\pi(N(\alpha_\lambda)) \cup \pi \circ f(C_0)\,)$.

Next, we show uniform bound of the energy $E(f)$. By definition 5.2, $\alpha_\lambda = p_\lambda^*(\beta\,\alpha)$ for some $\alpha \in \mathcal{H}$ and $p_\lambda^*(\alpha)$ is J-anti-invariant. We define C_- as the set of all z in C with $f^* p_\lambda^* \alpha(\,e_1(z), e_2(z)\,) \leq 0$, where $\{\,e_1(z), e_2(z) = j\,e_1(z)\,\}$ is an orthonormal basis of $T_z C$. Then (6.2) implies that $|\bar{\partial}_J f| \leq |\nu|$ on C_- and hence by (6.1) we have

$$0 \;\leq\; -f^* p_\lambda^* \alpha(\,e_1(z), e_2(z)\,) \;\leq\; 2|p_\lambda^* \alpha|\,|df|\,|\bar{\partial}_J f| \;\leq\; 2\,M\,|df|\,|\nu| \tag{6.6}$$

for any $z \in C_-$, where $M \,=\, \max\{\,|p_\lambda^* \alpha|\,|\,||\alpha_\lambda|| \leq N\,\}$. Therefore, we can conclude that

$$
\begin{aligned}
\frac{1}{2}\int_C |df|^2 \;&=\; \int_C |\bar{\partial}_J f|^2 \,+\, \omega(s + df) \\
&\leq\; \int_C f^* p_\lambda^*(\beta\alpha) + 2\int_C |df||\nu| + \omega(s + df) \\
&\leq\; \int_{C\setminus C_-} f^* p_\lambda^* \alpha + 2\int_C |df||\nu| + \omega(s + df) \\
&\leq\; -\int_{C_-} f^* p_\lambda^* \alpha + 2\int_C |df||\nu| + \omega(s + df) \\
&\leq\; (1 + 2M)\left(\int_C |\nu|^2\right)^{\frac{1}{2}}\left(\int_C |df|^2\right)^{\frac{1}{2}} + \omega(s + df)
\end{aligned}
$$

where the second inequality follows from (6.2), the fourth inequality follows from $p_\lambda^* \alpha\,(s + df) = 0$ and the last from (6.6). This implies the uniform energy bound independent of λ for sufficiently small $|\lambda|_\infty$. $\quad\square$

Remark 6.3 *Repeating the same argument as in section 4, one can show that the moduli spaces*

$$\overline{\mathcal{M}}_{g,k}(E(n), s + df, \mathcal{H}_{E(n)}, \nu) \quad and \quad \overline{\mathcal{M}}_{g,k}(Z_\lambda, s + df, \mathcal{H}_\lambda, \nu)$$

defines family invariants $GW_{g,k}(E(n), A, \mathcal{H}_{E(n)})$ and $GW_{g,k}(Z_\lambda, A, \mathcal{H}_\lambda)$, respectively. Moreover, by the standard corbodism argument as in Lemma 4.9 of [RT2] we have

$$GW_{g,k}(E(n), s + df, \mathcal{H}_{E(n)}) = GW_{g,k}^{\mathcal{H}}(E(n), s + df) = GW_{g,k}(Z_\lambda, s + df, \mathcal{H}_\lambda). \tag{6.7}$$

The following shows how maps into $Z_\lambda = E(n)$ split along the degeneration of $E(n)$. It is also a key observation for gluing of maps into $E(n)$ and $E(0)$, which leads to the sum formulas (2.5) and (2.6).

Lemma 6.4 *Let $\nu \in \mathcal{J}_0(Z)$ and $\{(f_\lambda, C_\lambda, \alpha_\lambda)\}$ be any sequence with $(f_\lambda, C_\lambda, \alpha_\lambda) \in \overline{\mathcal{M}}_{g,k}(\lambda, d)$. Then as $\lambda \to 0$, f_λ converges to a limit $f_0 : C_0 \to Z_0$ and α_λ converges to α_0, after passing to some subsequences, such that*

(a) the limit map f_0 can be decomposed as

$$f_1 : C_1 \to E(n), \quad f_2 : C_2 \to E(0), \quad and \quad f_3 : C_3 \to V$$

and f_1 (resp. f_2) represents homology class $s + d_1 f$ in $E(n)$ (resp. $s + d_2 f$ in $E(0)$) and f_3 represents $d_3[V]$ in V with $d_1 + d_2 + d_3 = d$,

(b) for $i = 1, 2$, each f_i transverse to V with $f_i^{-1}(V) = \{p_i\}$, where p_i is a node of C.

Proof. By Gromov Convergence Theorem and Lemma 6.2, f_λ converges to a limit $f_0 : C_0 \to Z_0$. Since $\alpha_\lambda = 0$ near $V \subset Z$, we have $J_{\alpha_\lambda} = J$ near V in Z. Therefore, (a) and (b) follows from Lemma 3.4 of [IP2] and Lemma 3.3 of [IP3].

7 Relative Invariants of $E(n)$

In this section, following [IP2], we define relative invariants of $E(n)$ relative to a smooth elliptic fiber $V = T^2$. In our case, the *rim tori* in $E(n) \setminus V$ disappear when we glue $E(n)$ and $E(0)$ along V. Together with the simple matching condition as in Lemma 6.4, that observation leads to the simple definition of relative invariants.

As in section 4, we fix the complex structure on $E(n)$. We also assume that we always work with a finite good cover p_μ as in (4.1) without specifying it. Throughout this section, A always denotes the class $s + df$.

For ν in $\mathcal{J}_0(Z)$, we define the relative moduli space by

$$\mathcal{M}_{g,k+1}^V(E(n), A, \mathcal{H}_{E(n)}, \nu) = \left\{ (f, \alpha) \in \overline{\mathcal{M}}_{g,k+1}(E(n), A, \mathcal{H}_{E(n)}, \nu) \mid f^{-1}(V) = \{x_{k+1}\} \right\}.$$

As in [IP2], we compactify this moduli space by taking its closure

$$C\mathcal{M}_{g,k+1}^V(E(n), A, \mathcal{H}_{E(n)}, \nu)$$

in the space of stable maps $\overline{\mathcal{M}}_{g,k+1}(E(n), A, \mathcal{H}_{E(n)}, \nu)$. Note that for each $\alpha \in \mathcal{H}_{E(n)}$, $\alpha = 0$ in some neighborhood of $V \subset E(n)$ and hence $J_\alpha = J$ on that neighborhood. Therefore, Proposition 7.1 below follows from the same arguments as in Lemma 4.2 and Proposition 6.1 of [IP2], and Theorem 4.2.

Proposition 7.1 *For generic ν in $\mathcal{J}_0(Z)$*

(a) $\mathcal{M}_{g,k+1}^V(E(n), A, \mathcal{H}_{E(n)}, \nu)$ is an orbifold of dimension $2 + 2(g+k)$ for $n = 0$ and $2(g+k)$ for $n \geq 1$, and

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(b) the Frontier of the map

$$\mathcal{M}^V_{g,k+1}(E(n),A,\mathcal{H}_{E(n)},\nu) \xrightarrow{\;st\times ev\times h\;} \overline{\mathcal{M}}_{g,k+1} \times E(n)^k \times V \qquad (7.1)$$

is contained in codimension at least 2, where ev is the evaluation map of the first k marked points and h is the evaluation map of the last marked point.

Proposition 7.1 together with Proposition 4.2 of [KM2] assert that the image of (7.1) gives rise to a rational homology class. We denote it by

$$[\,\mathcal{M}^V_{g,k+1}(E(n),A,\mathcal{H}_{E(n)})\,] \in H_*(\overline{\mathcal{M}}_{g,k+1};\mathbb{Q}) \otimes H_*(E(n)^k;\mathbb{Q}) \otimes H_*(V;\mathbb{Q}).$$

Definition 7.2 *For $2g + k \geq 3$, we define relative invariants by*

$$GW^V_{g,k+1}(E(n),A)(\,\beta;\alpha;C(\gamma)\,) = (\,\beta \otimes \alpha \otimes \gamma\,) \cap [\,\mathcal{M}^V_{g,k+1}(E(n),A,\mathcal{H}_{E(n)})\,]$$

where $\beta \in H^*(\overline{\mathcal{M}}_{g,k+1};\mathbb{Q})$, $\alpha \in H^*(E(n)^k;\mathbb{Q})$, and $\gamma \in H^*(V;\mathbb{Q})$.

The relative invariants of $E(0)$ and $E(1)$ defined as in Definition 7.2 are less finer than those in [IP2] (cf. Appendix in [IP3]). On the other hand, for another smooth fiber $U = T^2$ of $E(0)$ we can define relative invariants relative to both V and U as in Definition 7.2. In the below, we will denote ordinary and relative GW invariants for $E(0)$ by

$$\Phi_{A,g}, \quad \Phi^V_{A,g}, \quad \text{and} \quad \Phi^{V,U}_{A,g}, \quad \text{respectively.}$$

We end this section by relative invariants of $E(0)$ for the class $s + df$. Recall that for positive integer d, $\sigma(d)$ is the sum of the divisors, namely $\sigma(d) = \sum_{k|d} k$. For convenience we set $\sigma(0) = -1/24$.

Lemma 7.3 ([IP3]) *Let $V \subset E(0)$ be a smooth elliptic fiber.*

(a) $\Phi^V_{s+df,0}(\tau(f^*);C(f)) = 0$.

(b) $\Phi^V_{s+df,1}(\tau(f^*);C(pt)) = 2\sigma(d)$.

(c) $\Phi^V_{s+df,0}(pt;C(f)) = \Phi^V_{s+df,0}(C(pt)) = 1$ *if $d = 0$ and 0 otherwise.*

(d) $\Phi^V_{s+df,1}(pt;C(pt)) = d\,\sigma(d)$.

(e) $\Phi^{V,U}_{s+df,1}(C(pt),C(pt)) = 0$.

8 Sum Formula

This section shows the sum formulas (2.5) and (2.6) using a family version of Gluing Theorem — a map into Z_0 satisfying the matching condition as in Lemma 6.4 can be smoothed to produce a map into Z_λ. That smoothing relates invariants of $Z_\lambda = E(n)$ with relative invariants of $E(n)$ and $E(0)$ relative to a smooth fiber $V = T^2$.

Throughout this section we fix $n \geq 1$. Recall the evaluation map of last marked point as in (7.1). There is an evaluation map

$$ev_V : \bigcup (\, \mathcal{M}^V_{g_1,k_1+1}(\, E(n), s + d_1 f, \mathcal{H}_{E(n)}, \nu) \times \mathcal{M}^V_{g_2,k_2+1}(\, E(0), s + d_2 f, \nu)\,) \to V^2$$

which records the intersection points with V, where the union is over all $g_1 + g_2 = g$, $k_1 + k_2 = k$ and $d_1 + d_2 = d$. We set

$$\mathcal{M}^V_{g,k}(d) \;=\; ev_V^{-1}(\triangle) \tag{8.1}$$

where $\triangle$ is the diagonal of V^2. This space is an orbifold of dimension $2(g+k)$ for generic ν in $\mathcal{J}_0(Z)$ and comes with stabilization and evaluation maps

$$\mathcal{M}^V_{g,k}(d) \xrightarrow{\;st \times ev\;} (\overline{\mathcal{M}} \times E(n))_{g,k} \;=\; \bigcup (\overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1}) \times (E(n)^{k_1} \times E(0)^{k_2})$$

where the union is over all $g = g_1 + g_2$, and $k = k_1 + k_2$.

Now, consider a sequence of maps $(f_\lambda, \alpha_\lambda)$ in $\overline{\mathcal{M}}_{g,k}(\lambda, d)$. By Lemma 6.4, as $\lambda \to 0$, the maps $(f_\lambda, \alpha_\lambda)$ converge to a limit (f_0, α_0), after passing to some subsequences. In general, the limit (f_0, α_0) might be not in $\mathcal{M}^V_{g,k}(d)$. That happens if some components of f_0 map entirely into V. On the other hand, by Lemma 1.5 of [IP2] there is a constant c_V, depending only on (J_V, ν_V) such that every stable (J_V, ν_V)-holomorphic maps have energy great than c_V. This implies that for small $|\lambda|$ the energy of f_λ in the δ-neck

$$E^\delta(f_\lambda) \;=\; \frac{1}{2} \int |df_\lambda|^2 + |d\phi|^2 \tag{8.2}$$

is greater than c_V, where the integral is over $f_\lambda^{-1}(\, Z_\lambda(\delta)\,)$ and $\phi : C_\lambda \to \mathbb{P}^N$ as in (4.3). Therefore, for each λ if f_λ is δ-flat (see Definition 8.1 below) then f_0 is also δ-flat and hence the limit (f_0, α_0) is contained in $\mathcal{M}^V_{g,k}(d)$.

Following [IP3], we define δ-flat maps as follows:

Definition 8.1 *A stable (J, ν, α)-holomorphic map (f, α) into Z_λ is δ-flat if*

$$E^\delta(f) \;\leq\; \frac{c_V}{2} \tag{8.3}$$

Note that any δ-flat map (f, α) into Z_0 has no component maps into V. We denote by

$$\overline{\mathcal{M}}^\delta_{g,k}(\lambda, d) \;\subset\; \overline{\mathcal{M}}_{g,k}(\lambda, d) \quad (\text{ resp. } \mathcal{M}^{V,\delta}_{g,k}(d) \;\subset\; \mathcal{M}^V_{g,k}(d)\,) \tag{8.4}$$

the set of all δ-flat maps in $\overline{\mathcal{M}}_{g,k}(\lambda, d)$ (resp. in $\mathcal{M}^V_{g,k}(d)$).

The following is a family version of Theorem 10.1 of [IP3]. It shows that a δ-flat map into Z_0 can be smoothed to produce a δ-flat map into Z_λ for small $|\lambda|$.

Theorem 8.2 *For generic $\nu \in \mathcal{J}_0(Z)$ and for small $|\lambda|$, there is a diagram*

$$
\begin{array}{ccc}
\mathcal{M}_{g,k}^{V,\delta}(d) & \xrightarrow{\quad\Phi_\lambda\quad} & \overline{\mathcal{M}}_{g,k}^{\delta}(\lambda,d) \\
\downarrow{\scriptstyle st\times ev} & & \downarrow{\scriptstyle st\times ev} \\
\left(\overline{\mathcal{M}}\times E(n)\right)_{g,k} & & \overline{\mathcal{M}}_{g,k}\times Z_\lambda^k \\
& \searrow{\scriptstyle\sigma\times\pi_0} \quad {\scriptstyle id\times\pi_\lambda}\swarrow & \\
& \overline{\mathcal{M}}_{g,k}\times Z_0^k &
\end{array}
$$

which commutes up to homotopy, where Φ_λ is an embedding, σ is the gluing map of the domain as in (1.4), π_λ is the collapsing map as in (5.2), and $\pi_0 : \bigcup\left(E(n)^{k_1}\times E(0)^{k_2}\right) \to Z_0^k$ defined by $\pi_0(x_1,\cdots,x_{k_1},y_1,\cdots,y_{k_2}) = (x_1,\cdots,x_{k_1},y_1,\cdots,y_{k_2})$.

First, we use Theorem 8.2 to derive the sum formula (8.7) for certain constraints. Let $\beta = \beta_1 \otimes \cdots \otimes \beta_k \in H^{2r}(Z_0^k)$, where $r = g + k$. Denote by B^i a geometric representative of the Poincaré dual of β_i. We assume that for some $0 \le k_1 \le k$

(i) each B^i lies in $E(n)$-side if $i \le k_1$ and in $E(0)$-side if $i > k_1$, and

(ii) $\deg(\beta_1 \otimes \cdots \otimes \beta_{k_1}) = 2(g_1 + k_1)$ for some $0 \le g_1 \le g$.

Note that the assumption implies that there is a decomposition

$$
\pi_0^*\beta = \beta_1 + \beta_2 \quad\text{with}\quad \beta_1 \in H^{2(g_1+k_1)}\left(E(n)^{k_1}\right) \text{ and } \beta_2 \in H^{2(g_2+k_2)}\left(E(0)^{k_2}\right)
$$

where $g_2 = g - g_1$ and $k_2 = k - k_1$. On the other hand, the inverse image of B^i under π_λ gives a continuous family of geometric representatives B_λ^i of the Poincaré dual of $\pi_\lambda^*\beta_i$ in $H^*(Z_\lambda)$. We define the cut-down moduli spaces by

$$
\overline{\mathcal{M}}_{g,k}(\lambda,d) \cap \pi_\lambda^*\beta = \left\{(f,\alpha) \in \overline{\mathcal{M}}_{g,k}(\lambda,d) \mid ev_i(f,\alpha) \in B_\lambda^i\right\} \tag{8.5}
$$

$$
\mathcal{M}_{g,k}^V(d) \cap \pi_0^*\beta = \left\{((f_1,\alpha),f_2) \in \mathcal{M}_{g,k}^V(d) \mid ev_i((f_1,\alpha),f_2) \in B^i\right\} \tag{8.6}
$$

where ev_i is the evaluation map of the i-th marked point. Both cut-down moduli spaces (8.5) and (8.6) are finite. In particular, any maps in (8.5) is δ-flat for some $\delta > 0$.

Proposition 8.3 *Let $\beta \in H^{2r}(Z_0^k)$ be a constraint as above. Then*

$$
GW_{s+df,g}^{\mathcal{H}}(\beta) = \sum_{d_1+d_2=d} GW_{s+d_1f,g_1+1}^V(\beta_1;C(pt))\,\Phi_{s+d_2,g_2-1}^V(\beta_2;C(f))
$$

$$
+ \sum_{d_1+d_2=d} GW_{s+d_1f,g_1}^V(\beta_1;C(f))\,\Phi_{s+d_2f,g_2}^V(\beta_2;C(pt)). \tag{8.7}
$$

Proof. Denote the set of limits of sequences of maps in (8.5) as $\lambda \to 0$ by

$$
\lim_{\lambda\to 0}\left(\overline{\mathcal{M}}_{g,k}(\lambda,d) \cap \pi_\lambda^*\beta\right). \tag{8.8}
$$

We first assume that the limit set (8.8) is contained in the space (8.1). Then for small $|\lambda|$ all maps in (8.5) should be δ-flat. In that case Theorem 8.2 implies that

$$(id \times \pi_\lambda)_* [\overline{\mathcal{M}}_{g,k}(\lambda, d) \cap \pi_\lambda^* \beta] = (\sigma \times \pi_0)_* [\mathcal{M}_{g,k}^V(d) \cap \pi_0^* \beta] \qquad (8.9)$$

as a homology class in $H_0(\overline{\mathcal{M}}_{g,k} \times Z_0^k; \mathbb{Q})$. The left-hand side of (8.9) becomes

$$[\overline{\mathcal{M}}_{g,k}(\lambda, d) \cap \pi_\lambda^* \beta] = GW_{s+df,g}(Z_\lambda, H_\lambda)(\beta) = GW_{s+df,g}^{\mathcal{H}}(E(n))(\beta) \qquad (8.10)$$

where the second equality follows from (6.7). On the other hand, by assumption on the constraint β and the routine dimension count, the right-hand side of (8.9) becomes

$$
\begin{aligned}
[\mathcal{M}_{g,k}^V(d) \cap \pi_0^* \beta] = &\sum_{d_1+d_2=d} GW_{s+d_1 f, g_1+1}^V(\beta_1; C(pt)) \Phi_{s+d_2, g_2-1}^V(\beta_2; C(f)) \\
&+ \sum_{d_1+d_2=d} GW_{s+d_1 f, g_1}^V(\beta_1; C(f)) \Phi_{s+d_2 f, g_2}^V(\beta_2; C(pt)). \qquad (8.11)
\end{aligned}
$$

Therefore, if the limit set (8.8) is contained in the space (8.1) we have the sum formula (8.7) from (8.9), (8.11) and (8.10).

In general, the limit set (8.8) is not contained in the space (8.1). In that case, there are maps f_λ in (8.5) that converge to a limit f_0 as $\lambda \to 0$ such that some components of f_0 map entirely into V. The contribution of those maps in (8.5) is call the *contribution from the neck* and enters into the sum formula (8.7) as a correction term. This correction term can be computed by using the S-matrix (cf. section 12 of [IP3]). By the choice of the constraint β we have the correction term

$$\sum GW_{s+d_1 f, g_1}^V(\beta_1; C(f)) \Phi_{d_3 f, 1}^{V,U}(C(pt), C(pt)) \Phi_{s+d_2, g_2-1}^U(\beta_2; C(f)) \qquad (8.12)$$

where the sum is over all $d_1 + d_2 + d_3 = d$. The correction term (8.12) is zero by Lemma 7.3 e and hence the proof is complete. $\square$

Next, we use the sum formula (8.7) to compute relative invariants of $E(n)$.

Lemma 8.4 *Let γ_1, γ_2 be a basis of $H^1(E(0); Z)$.*

(a) $\Phi_{s+df,0}^V(\gamma_1, \gamma_1; C(f)) = 1$ *if $d = 0$ and 0 otherwise.*

(b) $\Phi_{s+df,1}^V(\gamma_1, \gamma_2; C(pt)) = 0$

(c) $GW_{s+df,g}^V(pt^{g-1}; C(pt)) = 0$

(d) $GW_{s+df,g}^V(pt^g; C(f)) = GW_{s+df,g}^{\mathcal{H}}(pt^g)$

Proof. (a) follows from (i) $\Phi_{s,0}(E(0))(\gamma_1, \gamma_2) = 1$ (see Theorem 2 of [LL]), (ii) for $g = 0$ relative invariants are same as absolute invariants (Proposition 14.9 of [IP3]), and (iii) there is no rational curve representing $s + df$ with $d \neq 0$ on $E(0) = S^2 \times T^2$ with a product complex structure

To prove (b), we will apply the sum formula (Theorem 12.4 of [IP3]) for the symplectic sum $E(0) = E(0) \#_V E(0)$. The only difference between that sum formula and (8.7) is the degree of constraints. We also note that as in the proof of Proposition 8.3 there is no contribution from the neck for our case.

Split constraints γ_1 and γ_2 on one side and one point constraint on the other side. Then we have

$$\Phi_{s+df,1}(\gamma_1, \gamma_2, pt) = \sum_{d_1+d_2=d} \Phi^V_{s+d_1 f,1}(\gamma_1, \gamma_2; C(pt)) \Phi^V_{s+d_2,0}(pt; C(f))$$

$$+ \sum_{d_1+d_2=d} \Phi^V_{s+d_1 f,0}(\gamma_1, \gamma_2; C(f)) \Phi^V_{s+d_2 f,1}(pt; C(pt)). \qquad (8.13)$$

Using Lemma 7.3 c,d, and (a), we can simplify (8.13) as

$$\Phi_{s+df,1}(\gamma_1, \gamma_2, pt) = \Phi^V_{s+df,0}(\gamma_1, \gamma_2; C(f)) + d\,\sigma(d). \qquad (8.14)$$

Now, (b) follows from (8.14) and $\Phi_{s+df,1}(pt, \gamma_1, \gamma_2) = d\,\sigma(d)$ (see Theorem 2 of [LL]).

Now, we use the sum formula (8.7), (a) and (b) to show (c) and (d). For the proof of (c), we split $g-1$ point constraints on $E(n)$-side and the constraints γ_1 and γ_2 on $E(0)$-side. Then we have

$$GW^{\mathcal{H}}_{s+df,g}(pt^{g-1}, \pi^*_\lambda \gamma_1, \pi^*_\lambda \gamma_2)$$

$$= \sum_{d_1+d_2=d} GW^V_{s+d_1 f,g}(pt^{g-1}; C(pt)) \Phi^V_{s+d_2,0}(\gamma_1, \gamma_2; C(f))$$

$$+ \sum_{d_1+d_2=d} GW^V_{s+d_1 f,g-1}(pt^{g-1}; C(f)) \Phi^V_{s+d_2 f,1}(\gamma_1, \gamma_2; C(pt)). \qquad (8.15)$$

Since $E(n)$ is simply connected, the left-hand side of (8.15) is zero. Therefore, (c) follows from (8.15) together with (a) and (b).

Lastly, we split g point constraints on $E(n)$-side to obtain

$$GW^{\mathcal{H}}_{s+df,g}(pt^g) = \sum_{d_1+d_2=d} GW^V_{s+d_1 f,g}(pt^g; C(f)) \Phi^V_{s+d_2 f,0}(C(pt)). \qquad (8.16)$$

Now, (d) follows from (8.16) and Lemma 7.3 c. $\qquad \square$

Finally, we are ready to show the sum formulas (2.5) and (2.6).

Proposition 8.5 (Sum Formulas)

(a) $H(t) = 2 F_0(t) \left(G(t) - \dfrac{1}{24} \right)$

(b) $F_g(t) = F_{g-1}(t)\, t\, G'(t)$

Proof. Choose a smooth fiber F on $E(0)$-side of Z_0 and consider the cut-down moduli space

$$\mathcal{M}^V_{1,1}(d) \cap \tau(f^*) = \{ (h, \alpha) \in \mathcal{M}^V_{1,1}(d) \mid ev(h) \in F \}.$$

The constraint $\tau(f^*)$ lies only on $E(0)$-side. We can thus apply the same argument as in the proof of Proposition 8.3 to obtain

$$GW^{\mathcal{H}}_{s+df,1}(\tau(f^*)) = \sum_{d_1+d_2=d} GW^V_{s+d_1 f,1}(C(pt)) \Phi^V_{s+d_2,0}(\tau(f^*); C(f))$$

$$+ \sum_{d_1+d_2=d} GW^V_{s+d_1 f,0}(C(f)) \Phi^V_{s+d_2 f,1}(\tau(f^*); C(pt)). \qquad (8.17)$$

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By Lemma 7.3 a,b and Lemma 8.4 d, (8.17) becomes

$$GW^{\mathcal{H}}_{s+df,1}(\tau(f^*)) \;=\; \sum_{d_1+d_2=d} 2\, GW^{\mathcal{H}}_{s+d_1f,0}\,\sigma(d_2). \tag{8.18}$$

Now, (a) follows from (8.18) and the definition of $F_0(t)$, $H(t)$, and $G(t)$.

To prove (b) we split $g-1$ points on $E(n)$-side and one point on $E(0)$-side. Then by (8.7) we have

$$\begin{aligned}
GW^{\mathcal{H}}_{s+df,g}(pt^g) \;=\; &\sum_{d_1+d_2=d} GW^{V}_{s+d_1f,g}(pt^{g-1};C(pt))\,\Phi^{V}_{s+d_2,0}(pt;C(f)) \\
&+\sum_{d_1+d_2=d} GW^{V}_{s+d_1f,g-1}(pt^{g-1};C(f))\,\Phi^{V}_{s+d_2f,1}(pt;C(pt)).
\end{aligned}$$

By Lemma 7.3 d and Lemma 8.4 c,d, this becomes

$$GW^{\mathcal{H}}_{s+df,g}(pt^g) \;=\; \sum_{d_1+d_2=d} GW^{\mathcal{H}}_{s+d_1f,g-1}(pt^{g-1})\,d_2\sigma(d_2). \tag{8.19}$$

Together with the definition of $F_g(t)$ and $G(t)$, (8.19) implies (b). $\qquad\square$

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